Determinants, Part 2
Lay 3.2

Oct. 9, 2013

So that we don't have to write "det" all the time, we will introduce a shorthand which Lay also uses. Instead of writing det and then square brackets, we just use vertical lines:

$$\det \begin{bmatrix} \dots \end{bmatrix} = \begin{vmatrix} \dots \end{vmatrix}.$$

1   Row Operations and $\det A$

We are first going to talk about some properties of the determinant.

Theorem 1.1. *Let A be a square matrix. Assume that B is a matrix produced from A by performing a single elementary row operation.*

- *If the row operation interchanged two rows, then $\det B = -\det A$;*

- *If the row operation added a multiple of one row to another, then $\det B = \det A$;*

- *if the row operation multiplied a row by r, then $\det B = r \det A$.*

Proof. Assume A is 2×2:
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Say we interchange the two rows of A. Then

$$\det B = bc - ad = -(ad - bc) = -\det A.$$

1

Assume that we add a multiple of one row to another. Say we added a multiple s of row 1 to row 2. Then

$$\det B = \det \begin{bmatrix} a & b \\ c - sa & d - sb \end{bmatrix} = a(d - sb) - b(c - sa)$$

$$= ad - sab + sab - bc = ad - bc = \det A.$$

If we added a multiple of row 2 to row 1 instead, the proof is similar. Finally, assume we multiply a row by some r. Then it is easy to see that

$$\det B = rad - rbc = r \det A.$$

In the case that A is bigger than 2×2, the proof proceeds by using cofactor expansion to reduce the properties of the $n \times n$ to the properties of the 2×2 determinant and then using the results we just proved in the 2×2 case. For details, see the end of Lay 3.2. $\qquad \square$

1.1 Efficiently computing $\det A$

Recall from last time that the determinant of a triangular matrix is just the product of the entries on the main diagonal of the matrix. One of the more efficient ways to compute a determinant of a general matrix A is to reduce A to a row echelon form B. Since you get from A to B using only row operations which add multiples of rows and row operations which interchange rows (recall the algorithm for row reduction does not multiply rows by constants until the end, when you are turning the echelon form into the RREF), we have

$$\det A = (-1)^p \det B,$$

where p is the number of row interchanges used in turning A into B. Since B is in echelon form, $\det B$ is just the product of the entries on the main diagonal of B.

Example 1.2. Let

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Then we compute a row echelon form B of the matrix A:

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 \\ 0 & -9 & -15 \\ 0 & -3 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 7 \\ 0 & -9 & -15 \\ 0 & 0 & -2 \end{bmatrix}.$$

We used no row interchanges in the production of this row echelon form. Therefore,

$$\det A = \begin{vmatrix} 1 & 5 & 7 \\ 0 & -9 & -15 \\ 0 & 0 & -2 \end{vmatrix} = 18.$$

This method of finding determinants is generally much faster than cofactor expansion. This is actually generally the algorithm used in computers for evaluating matrices, because of its efficiency. As an aside: one time when this method is not so useful is when the matrix is given in terms of variables. For instance, you may imagine a matrix A whose entries are polynomials in a variable x, and being asked to solve an equation like $\det A = 0$. Performing row reduction on a matrix whose entries are variables is quite difficult (you can try it if you don't believe me), so cofactor expansion is often the way to go.

Another way row operations help us compute determinants is by letting us "factor out" constants, which usually makes computing a row echelon form easier.

Example 1.3. Let

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 10 & 20 & 20 \\ -1 & -1 & 1 \end{bmatrix}.$$

Then

$$\det A = 3 \begin{vmatrix} 1 & 2 & 3 \\ 10 & 20 & 20 \\ -1 & -1 & 1 \end{vmatrix} = 3 \cdot 10 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix}.$$

In this form, it is much easier to compute a row echelon form.

$$\det A = 30 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix} = 30 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 4 \end{vmatrix} = -30 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{vmatrix},$$

where in the last step we have used a row interchange. Therefore, $\det A = (-30)(1)(1)(-1) = 30$.

2 Invertibility and $\det$

The connection to invertibility is now clear.

Theorem 2.1. *Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det A \neq 0$.*

Proof. A is invertible if and only if it has n pivot positions. This happens if and only if A has a row echelon form with nonzero entries on the main diagonal. By what we know about determinants of triangular matrices and the behavior of det under row operations, A has a row echelon form with nonzero entries on the main diagonal if and only if $\det A \neq 0$. $\qquad\square$

This theorem gives us another item to tack onto the list of the Invertible Matrix Theorem.

3 More properties

Here it will actually be useful to think about the transpose a bit (go recall the definition of the transpose if you have forgotten).

Theorem 3.1. *If A is an $n \times n$ matrix, then $\det A = \det A^T$.*

Proof. The theorem is clearly true if A is a 1×1 matrix, because then $A = A^T$. We now perform "induction". That is, we assume that the theorem holds for all values of n up to some number k, then show that this means it must also hold when $n = k + 1$.

So let A be a $(k + 1) \times (k + 1)$ matrix. Then

$$\det A = \sum_{j=1}^{k+1} a_{1j}(-1)^{1+j} \det A_{1j}. \tag{1}$$

Now, if we denote the matrix produced by removing row i and column k from A^T by A_{ik}^T, note that $A_{ik}^T = (A^T)_{ki}$. Also, note that A_{ik} is a $k \times k$ matrix, so its determinant is equal to the determinant of its transpose. Applying this

4

in (1) gives

$$\det A = \sum_{j=1}^{k+1} a_{1j}(-1)^{1+j} \det(A_{j1})^T$$

$$= \sum_{j=1}^{k+1} a_{1j}(-1)^{1+j} \det(A^T)_{j1}$$

$$= \sum_{j=1}^{k+1} a_{1j}(-1)^{1+j} \det A_{1j}^T$$

$$= \det A^T,$$

because this is the cofactor expansion of A^T down the first column. □

What this theorem means is that if we define **elementary column operations** the same way as row operations– add a multiple of one column to another, interchange columns, etc– then these affect the determinant in the same way as the corresponding row operations.

3.1   Multiplication and linearity

The following theorem will be presented without proof:

Theorem 3.2. *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

The standard proof of this theorem uses a clever argument involving elementary matrices; see Lay for a proof.

Last, we describe the multilinearity of the determinant. Let

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$$

be a square matrix. Pick some column–say, the jth column– of A and define a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as the determinant of the matrix obtained by replacing column j with the argument of f: that is,

$$f(\mathbf{x}) = \det \begin{bmatrix} \mathbf{a}_1 & \dots \mathbf{a}_{j-1} & \mathbf{x} & \dots & \mathbf{a}_n \end{bmatrix}.$$

Then f is a linear function. That is, for all scalars c and all vectors $\mathbf{x}, \mathbf{y}$, we have $f(c\mathbf{x}) = cf(\mathbf{x})$ and $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$.